Supporting Information for

**Parametric amplification and self-oscillation in a nanotube mechanical resonator**

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Equations 3 and 4 in the main text are drawn from reference [1]. Please note that we use a different notation than in [1] in order to reserve certain symbols for physical quantities.

**A) Derivation of equation 2**

We apply $V_g^{AC}(t) = V_g^{AC}\cos(\omega t)$ with $\omega = 2\pi f$ . The resulting force $F\cos(\omega t)$ induces an oscillation of the nanotube position $z = \mathrm{Re}[\tilde{z}(\omega)]\cos(\omega t) + \mathrm{Im}[\tilde{z}(\omega)]\sin(\omega t)$ with

$$\tilde{z}(\omega) = \pi F / m(\omega_0^2 - \omega^2 - i\omega_0\omega/Q) + \pi F / m(\omega_0^2 - \omega^2 + i\omega_0\omega/Q) \qquad \text{(S0)}$$

and an oscillation of the nanotube conductance [2]

$$\delta G = \frac{\partial G}{\partial V_g}\left(V_g^{AC}\cos(\omega t) + zV_g^{DC}\frac{C_g'}{C_g}\right). \qquad \text{(S1)}$$

When applying $V_{sd}^{AC}(t) = V_{sd}^{AC}\cos((\omega - \delta\omega)t + \varphi_E)$ , we get

$$I_{mix} = \frac{1}{2}V_{sd}^{AC}\frac{\partial G}{\partial V_g}\left(V_g^{AC}\cos(\delta\omega\cdot t - \varphi_E) + V_g^{DC}\frac{C_g'}{C_g}\cos(\delta\omega\cdot t - \varphi_E)\mathrm{Re}[\tilde{z}(\omega)] + V_g^{DC}\frac{C_g'}{C_g}\sin(\delta\omega\cdot t - \varphi_E)\mathrm{Im}[\tilde{z}(\omega)]\right) \qquad \text{(S2)}$$

at frequency $\delta\omega$ .

**B) Derivation of equation 3**

As we explain in the main text, we tune the phase of the lock-in amplifier with which we measure the mixing current, such that $X \propto \mathrm{Re}[\tilde{z}(\omega)]$ and $Y \propto \mathrm{Im}[\tilde{z}(\omega)]$ . The secular perturbation theory in [1] employs dimensionless variables that are related to the physical ones by

$$\xi = z\sqrt{\frac{\alpha}{m\omega_0^2}}\,; \qquad G = \frac{F}{\omega_0^3}\sqrt{\frac{\alpha}{m^3}}\,; \qquad \bar{t} = \omega_0 t\,; \qquad \text{and } \bar{\omega} = \frac{\omega}{\omega_0}\,; \qquad \text{(S3)}$$

where $\alpha$ denotes the coefficient of the Duffing cubic force, $m$ the resonator mass, $F$ the coefficient of the driving force $F\cos(\omega t)$ , and $\omega_0 = 2\pi f_0$ . The other variables are defined in the main text.

In a next step, a complex amplitude $A(T)$ is introduced, where $T = \varepsilon \cdot \bar{t}$ is a slow time variable and $\varepsilon = 1/Q_0$ ( $Q_0 = m\omega_0/\gamma$ is the quality factor, $\gamma$ being the linear damping constant). Following [1] we use the ansatz

$$\xi(\bar{t}) = \frac{\sqrt{\varepsilon}}{2}\left(A(T)\cdot e^{i\cdot\bar{t}} + c.c.\right), \tag{S4}$$

where $c.c.$ denotes complex conjugation. Assuming a steady-state solution of the form

$$A(T) = ae^{i\Omega T} = |a|e^{i\phi}e^{i\Omega T} \tag{S5}$$

this leads to the expressions

$$\xi(\bar{t}) = |a|\sqrt{\varepsilon}\cos(\overline{\omega}\cdot\bar{t} + \phi) \tag{S6}$$

$$z(t) = |a|\sqrt{\gamma\omega_0/\alpha}\cos(\omega t + \phi)\quad . \tag{S7}$$

Using $a = \mathrm{Re}[a] + i\,\mathrm{Im}[a]$ and $e^{i(\Omega T+\bar{t})} = \cos(\Omega T + \bar{t}) + i\sin(\Omega T + \bar{t})$, we get that

$$z(t) = \sqrt{\gamma\omega_0/\alpha}\left(\mathrm{Re}[a]\cos(\omega t) - \mathrm{Im}[a]\sin(\omega t)\right) \tag{S8}$$

Without pumping, we have at resonance (defined as the frequency for which the motional amplitude is largest) $\mathrm{Re}[a] = 0$ and $|\mathrm{Im}[a]| = |g|$ where $g = G\varepsilon^{-3/2}$ (using eq. (1.30) of [1] and assuming that the nonlinear damping force is negligible), so

$$X_{unpumped} = 0 \text{ and } Y_{unpumped} = r\cdot g \tag{S9}$$

with $r$ a real constant (using eq. S2 and S8).

When the pumping is on (i.e. the spring constant is modulated as $k(1 + H\cos(\omega_p t))$), eq. (1.52) of [1] reads

$$a = -e^{i\pi/4}\left(\frac{\cos(\Delta\phi + \pi/4)}{1 - h/2} + i\frac{\sin(\Delta\phi + \pi/4)}{1 + h/2}\right)|g| \tag{S10}$$

where $\Delta\phi$ is the phase of the driving force with respect to the pumping and $h/2 = V_P/V_{P,C}$ (here $h = H/\varepsilon = \frac{2Q_0}{f_0}\frac{df_0}{dV_g}V_P$ and $V_{P,C} = (f_0\cdot dV_g/df_0)/Q_0$ ). Please note that the equation appears in [1] without a minus sign. We measure at resonance

$$Y = r\cdot\mathrm{Im}[a]\quad . \tag{S11}$$

Using eq. S9, S10, and S11, we obtain

$$\left|\frac{Y_{pumped}}{Y_{unpumped}}\right| = \left|\mathrm{Im}\left[-e^{i\pi/4}\left(\frac{\cos(\Delta\phi + \pi/4)}{1 - V_P/V_{P,C}} + i\frac{\sin(\Delta\phi + \pi/4)}{1 + V_P/V_{P,C}}\right)\right]\right| \tag{S11}$$

**C) Derivation of equation 4**

Introducing the nonlinear damping force $\eta z^2\dot{z}$ in the Newton equation, Lifshitz and Cross obtained (eq. 1.70 of [1])

$$\frac{db}{dT} = \frac{1}{2}\frac{h - h_C}{h_C}b - \frac{\sigma}{8}b^3 + \frac{|g|}{2}\cos(\Delta\phi + \pi/4) \tag{S12}$$

where $b = Ae^{-i\pi/4}$ is a real constant, $\sigma = \frac{\eta\omega_0}{\alpha}$, and $h_C = \frac{2Q_0}{f_0}\frac{df_0}{dV_g}V_{P,C}$. Please note that the last term on the right-hand side has a different sign in [1] (because of the minus sign in eq. S10). Following [1], we are interested in a time-independent solution ($db/dT = 0$) at maximum gain ($\Delta\phi = -\pi/4$). At resonance, we have $\Omega = 0$. We require a solution for $\mathrm{Im}[a] = \mathrm{Im}[A] = b/\sqrt{2}$, which satisfies

$$h = \frac{1}{2}h_C\sigma\,\mathrm{Im}[a]^2 - \frac{1}{\sqrt{2}}\frac{|g|h_C}{\mathrm{Im}[a]} + h_C. \tag{S13}$$

After inserting the physical units and using eq. S3 and S8, we get

$$V_P = \frac{\pi\eta Q_0 f_0 V_{P,C}}{k_0}\mathrm{Im}[\tilde{z}(\omega)]^2 - \frac{1}{\sqrt{2}}\frac{Q_0 F V_{P,C}}{k_0}\frac{1}{\mathrm{Im}[\tilde{z}(\omega)]} + V_{P,C}, \tag{S14}$$

which we simplify to

$$V_P = u\Lambda^2 - \frac{v}{\Lambda} + V_{P,C} \tag{S15}$$

with $u$, $v$, and $V_{P,C}$ as fitting parameters. Here, we make use of the relations $\Lambda = \left|Y_{pumped}/Y_{unpumped}\right|$ and $Y \propto \mathrm{Im}[\tilde{z}(\omega)]$ to write

$$\mathrm{Im}[\tilde{z}(\omega)] = \Lambda \cdot \mathrm{Im}[\tilde{z}(\omega)]_{unpumed} \tag{S16}$$

so that $u = \frac{\pi\eta Q_0 f_0 V_{P,C}}{k_0}\mathrm{Im}[\tilde{z}(\omega)]^2_{unpumped}$ and $v = \frac{Q_0 F V_{P,C}}{\sqrt{2}k_0\,\mathrm{Im}[\tilde{z}(\omega)]_{unpumped}}$. Please note that the value of $V_{P,C}$ is independent of any renormalization of the motion amplitude.

**D) Additional measurements of self-oscillations.**

We present self-oscillation measurements at $300$ K in Fig. S1. The device is the same as that in the main text, but measured at a time when mechanical and electrical characteristics were different: namely, the conductance is larger by $20$ % and $df_0/dV_g$ of the first mechanical mode is higher ($7$ MHz/V). In addition, the measurements are performed at a different gate voltage ($-1.9$ V). The quality factor obtained from the self-oscillation threshold is $\sim 230$. This is much larger then the quality factor determined from the lineshape of the driven resonance, which is $10$-$15$ (Fig. S1b).

Figure S2 shows measurements from a second carbon nanotube resonator. The quality factor obtained from the self-oscillation threshold is $\sim 1000$. This is again much larger than the quality factor determined from the lineshape of the driven resonance, which is about $100$-$220$ (Fig. S2c).

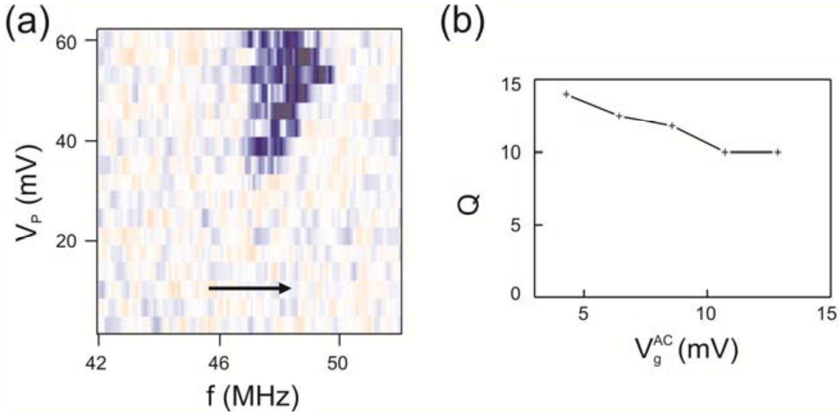


**Figure S1: (a)** Self-oscillations at $300$ K and with $V_g = -1.9$ V. Here, $f_0 = 48$ MHz, $df_0/dV_g = 7$ MHz/V, and $V_{P,C} \sim 30$ mV. The corresponding quality factor is $\sim 230$. **(b)** Quality factor as a function of the driving voltage $V_g^{AC}$ in the absence of parametric pumping, obtained by fitting the resonance lineshape with the predictions of a damped harmonic oscillator.

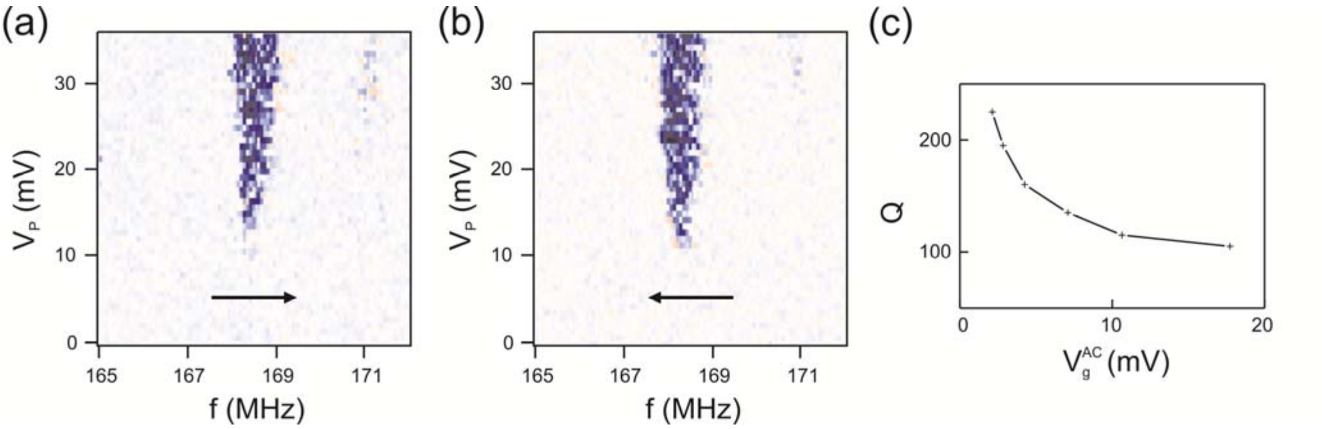


**Figure S2:** Data from a second nanotube device at $T = 60$ K and $V_g = 1.8$ V **(a) and (b)** Self-oscillation with increasing and decreasing frequency sweeps, respectively. Self-oscillations are detected above $V_{P,C} = 10$ mV in a tongue-shaped region, which corresponds to a quality factor of $\sim 1000$ ( $f_0 \sim 168$ MHz and $df_0/dV_g = 14.2$ MHz/V). In contrast to the data shown in Fig. 3 of the main text, no hysteresis is observed. **(c)** Quality factor as a function of the driving voltage $V_g^{AC}$ in the absence of parametric pumping, obtained by fitting the resonance lineshape with the predictions of a damped harmonic oscillator.

**References**

1. Lifshitz, R.; Cross, M. C. *Reviews of Nonlinear Dynamics and Complexity*, Wiley-VCH: New York, 2008; Vol. 1 [www.tau.ac.il/~ronlif/pubs/RNDC1-1-2008-preprint.pdf]

2. Sazanova, V.; Yaish, Y.; Üstünel, H.; Roundy, D.; Arias, T. A.; McEuen, P. L. *Nature* **2004,** *431,* 284